Journal of the
ENGINEERING MECHANICS DIVISION
Proceedings of the American Society of Civil Engineers

ELASTIC LOADING OF HIGH PRESSURE CYLINDERS

By Raymond Epain[1] and Boris Vodar[2]

SYNOPSIS

A thick-walled cylinder submitted to uniformly distributed internal and external pressures and to a uniformly distributed longitudinal load is considered. A graphical construction is established allowing the determination of whether the material does or does not remain elastic under this state of loads, or the selection of the value of one pressure with a view to maximizing another without the cylinder undergoing plastic deformation. Three different constructions are given corresponding to the use of the criteria of Von Mises, Tresca, and of a linearized form of the intrinsic curve of Mohr-Caquot. Several remarks on the conditions and limits in the use of this method are included.

INTRODUCTION

Notation.—The letter symbols adopted for use in this paper are defined where they first appear and are listed alphabetically in the Appendix.

Note.—Discussion open until March 1, 1965. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 90, No. EM5, October, 1964.

1 Engineer, Laboratoire des Hautes Pressiers, Centre National De La Recherche Scientifique, Bellevue, France.

2 Director, Laboratoire des Hautes Pressiers, Centre National De La Recherche Scientifique, Bellevue, France.

A hollow cylinder of circular cross-section, Fig. 1, is submitted to internal, p_i, external, p_e and longitudinal, p_l uniformly distributed pressures, and the limiting relations, i.e., the limit within which the vessel undergoes no plastic deformation between these quantities, are established. These relations will be referred to as elastic loading conditions and will be established for three criteria of plasticity:—the criteria of Von Mises,[3] of Mohr-Caquot[4] and of Tresca.[5]

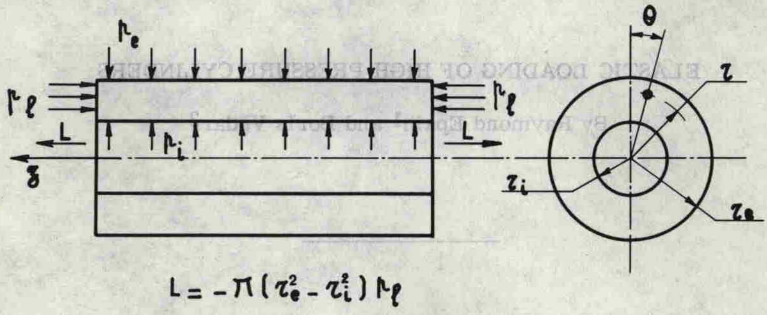

FIG. 1

The well-known formulas of Lamé[6] give the radial, σ_r, circumferential, σ_θ, and longitudinal, σ_z, stresses as functions of p_i, p_e and p_l in the following form:

$$\sigma_r = p_i \frac{1}{k^2 - 1}\left[1 - \frac{r_i^2}{r^2} k^2\right] + p_e \frac{k^2}{k^2 - 1}\left[\frac{r_i^2}{r^2} - 1\right] \quad \text{..} \quad (1)$$

$$\sigma_\theta = p_i \frac{1}{k^2 - 1}\left[1 + \frac{r_i^2}{r^2} k^2\right] - p_e \frac{k^2}{k^2 - 1}\left[\frac{r_i^2}{r^2} + 1\right] \quad \text{..} \quad (2)$$

and

$$\sigma_z = - p_l \quad \text{..................} \quad (3)$$

[3] Von Mises, R., "Mechanik der festen Körper im plastisch deformablen Zustand," Göttinger Nachrichten, 1913.

[4] Caquot, A., "Définition du domaine élastique dans les corps isotropes," Proceedings, 4th Congress of Internatl. Applied Mechanics, Cambridge, Mass., 1935, p. 24.

[5] Tresca, H. E., "Mémoire sur l'écoulement des corps solides," Mémoires présentés par divers savants, Vol. 18, 1868, pp. 773-799.

[6] Lamé, G., et Clapeyron, B. P., "Mémoires sur l'équilibre intérieur des corps solides homogènes," Mémoires présentés par divers savants, 1833.

in which

$$k = \frac{r_e}{r_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

is the ratio of the external radius to the internal radius of the cylinder. By substituting Eqs. 1, 2, and 3 in the corresponding criterion, the desired conditions of elastic loading are obtained.

ELASTIC LOADING FOR VON MISES' CRITERION

The substitution of Eqs. 1, 2, and 3 in the relation of Von Mises, given by

$$\left(\sigma_r - \sigma_\theta\right)^2 + \left(\sigma_\theta - \sigma_z\right)^2 + \left(\sigma_z - \sigma_r\right)^2 < 2\,\sigma_0^{\;2} \quad \ldots. \quad (5)$$

yields the condition of elastic loading corresponding to this criterion

$$\frac{2}{\left(k^2 - 1\right)^2}\left[3\,k^4 \frac{r_i^4}{r^4}\left(p_i - p_e\right)^2 + p_i^2 + k^4\,p_e^2 - 2\,k^2\,p_i\,p_e + p_l^2\left(k^2 - 1\right)^2 + 2\,p_i\,p_l\left(k^2 - 1\right) - 2\,p_e\,p_l\,k^2\left(k^2 - 1\right)\right] < 2\,\sigma_0^2 \quad \ldots \quad (6)$$

in which σ_0 is the elastic limit of the material for pure tension. The left side of Eq. 6 is a maximum for $r = r_i$, and demonstrates that plastic deformations will occur, either first at the internal diameter of the cylinder whatever the relative values of p_i and p_e, or simultaneously in the entire thickness of the cylinder for the particular case $p_i = p_e = 0$ and $p_l = \pm\,\sigma_0$.

Now, consider the case in which a plastic deformation is possible. The relation, $r = r_i$ is written, the inequality[7] in Eq. 6 becomes equality. On the pressure space p_i, p_l, p_e, the surface described by this equality is an elliptic cylinder[7] with its axis pointing in the direction (1, 1, 1). The elliptic cross-section varies both in dimension and orientation, with k. This surface has meaning only as long as p_i and p_e are positive, while

$$p_l = -\,\frac{L}{\pi\left(r_e^2 - r_i^2\right)} \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

may be positive, negative or vanishing, depending on the value and sign (tension or compression) of the longitudinal load L.

This surface is studied in the system of orthonormal axes V, W, Z, with V and W being respectively coincident with the minor and major axes of the ellipse

[7] Epain, R., "Contribution à l'étude de la résistance des cylindres épais elasto-plastiques," thesis presented to the University of Paris, at Paris, France, in 1961, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

of the normal cross section, while Z, parallel to the generating line of the elliptic cylinder, is inclined at equal angles to p_i, p_l, p_e.

The dimensions of the ellipse of the normal cross section are given by

$$\frac{\text{minor axis}}{2} = \frac{\sigma_0\,(M-1)}{\sqrt{4\,M^2 - M + 1 + \sqrt{7\,M^4 + 10\,M^3 - 2\,M + 1}}} \quad \ldots \quad (8)$$

and

$$\frac{\text{major axis}}{2} = \frac{\sigma_0\,(M-1)}{\sqrt{4\,M^2 - M + 1 - \sqrt{7\,M^4 + 10\,M^3 - 2\,M + 1}}} \quad \ldots \quad (9)$$

while its orientation, relative to the projections p_i', p_l', p_e' of the axes p_i, p_l, p_e onto the plane, π, perpendicular to Z, is determined by

$$\tan\psi = \sqrt{\frac{\left(\dfrac{V_1}{V_3}\right)^2 + \left(\dfrac{V_2}{V_3}\right)^2 + 1}{\left(\dfrac{W_1}{W_3}\right)^2 + \left(\dfrac{W_2}{W_3}\right)^2 + 1}} \quad \ldots \ldots \ldots \quad (10a)$$

in which

$$\psi = \text{angle } V, p_e' \quad \ldots \ldots \ldots \quad (10b)$$

$$\frac{V_1}{V_3} = \frac{3\,M^2\,(M-1) + \chi_1}{3\,M^2\,(M-1) - M\,\chi_1} \quad \ldots \ldots \ldots \quad (10c)$$

$$\frac{W_1}{W_3} = \frac{3\,M^2\,(M-1) + \chi_2}{3\,M^2\,(M-1) - M\,\chi_2} \quad \ldots \ldots \ldots \quad (10d)$$

$$\frac{V_2}{V_3} = \frac{3\,M^2\,(M-1)^2 + (M-1)\,\chi_1}{3\,M^2\,(M-1)^2 - (3\,M+1)\,M\,\chi_1} \quad \ldots \ldots \ldots \quad (10e)$$

$$\frac{W_2}{W_3} = \frac{3\,M^2\,(M-1)^2 + (M-1)\,\chi_2}{3\,M^2\,(M-1)^2 - (3\,M+1)\,M\,\chi_2} \quad \ldots\ldots \quad (10f)$$

$$\left.\begin{matrix}\chi_1\\ \chi_2\end{matrix}\right| = 4\,M^2 - M + 1 \pm \sqrt{7\,M^4 + 10\,M^3 - 2\,M + 1} \quad \ldots \quad (10g)$$

and

$$M = k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10h)$$

For k = 1, corresponding to a hypothetical cylinder of zero thickness, the ellipse is reduced to a line (zero surface) along p_l'. For $k = \infty$ corresponding

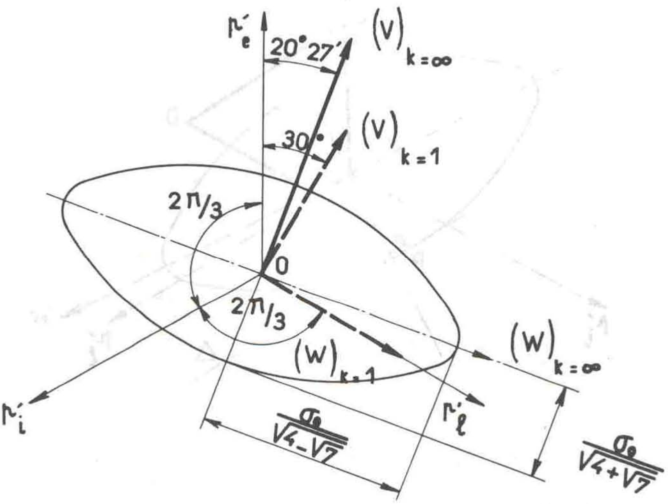

FIG. 2

to a cylinder of infinite thickness or to a capillary tube, the ellipse has the dimensions indicated in Fig. 2.

If, in the pressure space, the load is represented by the vector, OP, with components p_i, p_l, p_e, the following remarks can be made:

1. Plastic flow is only possible if P lies on the elliptic cylinder;
2. since hydrostatic load is represented by a vector parallel to Z, only the component of $\overrightarrow{OP}$ in the plane, π, is necessary for determining whether the material does or does not remain elastic; and

3. the ellipse corresponding to k = ∞ having a finite dimension confirm the known result that a finite state of load is sufficient to create a plastic deformation in a cylinder of infinite thickness.

The second statement leads to the establishment of a graphic method permitting the resolution of the problems relative to elastic loading. On a first graph, A, three equidistant axes p'_i, p'_l, p'_e are traced, as well as the axes, V, for different values of K. For the same values a series of graphs, B, are traced on transparent paper representing the corresponding ellipses. The number of these graphs is limited both by the allowed interpolations and the fact that k = 4 constitutes a limiting value in practice. Finally a new simpli-

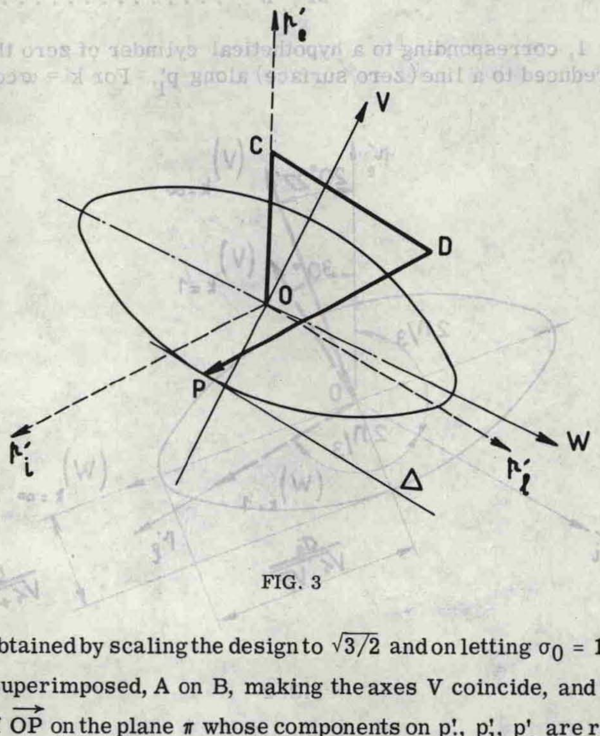

FIG. 3

fication is obtained by scaling the design to $\sqrt{3/2}$ and on letting $\sigma_0 = 1$. Then the graphs are superimposed, A on B, making the axes V coincide, and tracing the projection of $\overrightarrow{OP}$ on the plane π whose components on p'_i, p'_l, p'_e are respectively p_i/σ_0, p_l/σ_0, p_e/σ_0. The cylinder does or does not remain elastic according to whether P falls inside or on the ellipse.

With this method it is possible to find graphically, for a given value of one of the three variables, the possible maximum of one of the other two and the corresponding value of the third. Some of these results are well known. For example, it is shown in Fig. 3, how for a given value of p_e, p_l could be determined so that p_i is a maximum. Beginning as before, $\overline{OC} = p_e/\sigma_0$ and Δ are traced parallel to p_1' and tangent to the ellipse. It is then deduced that $p_1 = \sigma_0$ CD and $p_i = \sigma_0$ DP.

The determination of the extrema can be done by this method, thus producing the following results:

Maximum Value	Given Value	Corresponding Value	
$p_i = p_e + \frac{\sigma_0}{\sqrt{3}}\left(1 - \frac{1}{k^2}\right)$	p_e	$p_1 = p_e - \frac{\sigma_0}{\sqrt{3}\,k^2}$	(11a)
$p_i = \frac{2\,\sigma_0}{\sqrt{3}} + p_e$	p_1	$p_e = \frac{\sigma_0}{2\sqrt{3}}\left(3 + \frac{1}{k^2}\right) + p_1$ ..	(11b)
$p_e = p_1 + \sigma_0\sqrt{1 + \frac{1}{3\,k^4}}$	p_1	$p_i = p_e + \frac{\sigma_0}{\sqrt{3}}\,\frac{3\,k^2 + 1}{\sqrt{3\,k^4 + 1}}$ ..	(11c)
$p_e = p_i + \frac{\sigma_0}{\sqrt{3}}\left(1 - \frac{1}{k^2}\right)$	p_i	$p_1 = p_i + \frac{\sigma_0}{\sqrt{3}}$	(11d)
$p_1 = p_i + \frac{2\,\sigma_0}{\sqrt{3}}$	p_i	$p_e = p_i + \frac{\sigma_0}{2\sqrt{3}}\left(1 - \frac{1}{k^2}\right)$ ..	(11e)

and

$p_1 = p_e + \sigma_0\sqrt{1 + \frac{1}{3\,k^4}}$	p_e	$p_i = \dfrac{p_e - \frac{\sigma_0}{\sqrt{3}}\left(1 - \frac{1}{k^2}\right)}{\sqrt{3\,k^4 + 1}}$ ..	(11f)

ELASTIC LOADING FOR THE CRITERION OF THE INTRINSIC CURVE OF MOHR-CAQUOT

For simplification, a linearized intrinsic curve is used, obtained by drawing the right lines tangents to the circles of diameters, σ_0 and σ_c, in which σ_0 and σ_c are the absolute values of the elastic limits for pure tension and pure compression.

There is plastic flow at a point in the wall of the cylinder if the local values of the constraints are such that the Mohr circle constructed by the major σ_M and the minor σ_m stressed is tangent to or cuts these lines and the necessary condition that the cylinder remains elastic is expressed by the inequality

$$\sigma_M - \sigma_m \frac{\sigma_0}{\sigma_c} < \sigma_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

According to the relative magnitudes of the principal stresses given by Eqs. 1, 2, and 3, Eq. 12 can be written in six different ways. Just as for the criterion of Von Mises, it can be shown that plastic flow begins at the internal diameter. Then, at the limit, and with $r = r_i$, these inequalities become equalities and present six planes in the space p_i, p_e, p_l.

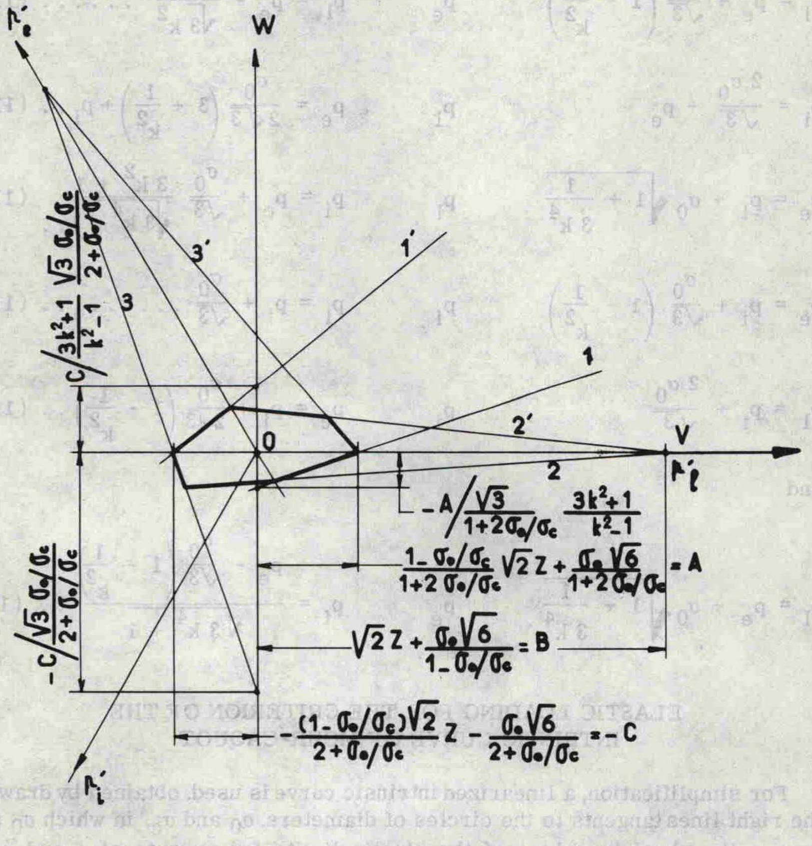

FIG. 4

In the new coordinate system V, W, Z (where the axis Z coincides with the line $p_i = p_e = p_l$ while W is at the intersection of the plane formed by the coordinates p_i and p_e and the plane π perpendicular to Z and passing through the origin) the contours formed by their traces on the planes perpendicular to the line $p_i = p_l = p_e$ are represented in Fig. 4. It can be seen that the slopes of these traces vary with the ratios σ_0/σ_c and k (except for the lines 3 and 3').

Furthermore, their ordinate intersections are function of σ_0/σ_c and in particular, of

$$Z_o = \frac{1}{\sqrt{3}}\left(p_i + p_l + p_e\right) \quad \ldots\ldots\ldots\ldots \quad (13)$$

This signifies that, contrary to the criterion of Von Mises, the extent of the elastic domain in the present case is no longer independent of the hydrostatic component of the load vector; the elastic domain enlarges as Z increase, the lines forming the contour remaining parallel to themselves. Moreover, it should be noted that, with the system of axes V, W, Z, the V-axis coincides with the projection p_l' of p_l on the plane π.

The graphic method described in the preceding paragraph is also valid in this case. It is slightly complicated because the dimensions of the elastic zone

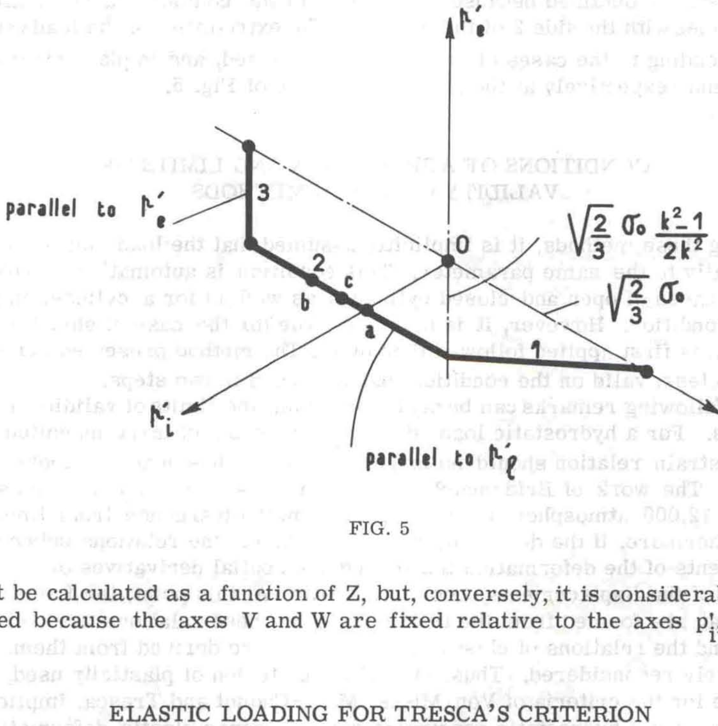

FIG. 5

must be calculated as a function of Z, but, conversely, it is considerably simplified because the axes V and W are fixed relative to the axes p_i', p_l', p_e'.

ELASTIC LOADING FOR TRESCA'S CRITERION

Letting $\sigma_0 = \sigma_c$, the oblique lines of the precedent intrinsic curve become parallel and the criterion of Mohr-Caquot reduces to that of Tresca. In the space p_i, p_l, p_e, the criterion of Tresca is represented by an irregular hexagonal prism inscribed in the elliptic cylinder of Von Mises. The magnitude of the elastic domain is once again, as in the case of Von Mises, independent of the hydrostatic component of the load vector. The intersection of this prism with the plane π gives the contour of the elastic domain. Fig. 5 shows that this contour can be easily traced, the other half of the hexagon being symmetric with respect to the origin. For $k = 3$, side 1 of the hexagon almost coincides with the perpendicular to p'_e (the position of 1 in the figure corresponding to $k = \infty$). This shows that increasing k above the value 3 adds only a small gain to the elastic loading. For $k = 1$, the hexagon is reduced to the line p'_l (zero surface).

The study of the maxima uncovers the following well-known result: For a given p_e there exists, contrary to the criterion of Von Mises, an infinite number of values of p_l for which p_i is maximum and equal to

$$\frac{\sigma_0}{2}\left(1 - \frac{1}{k^2}\right)$$

This result is obtained because the tangent of the contour drawn parallel to p'_l coincides with the side 2 of the hexagon. The extremities of the load vectors corresponding to the cases of cylinders open, closed, and in plane strain condition, end respectively at the points a, b, and c of Fig. 5.

CONDITIONS OF APPLICATION AND LIMITS OF VALIDITY OF THESE METHODS

Using these methods, it is implicitly assumed that the loads increase proportionally to the same parameter. This condition is automatically satisfied for the cases of open and closed cylinders as well as for a cylinder in plane strain condition. However, it is no longer true for the case of shrink fits, in which p_e is first applied followed then by p_i. The method presented herein is, nevertheless, valid on the condition that it is used in two steps.

The following remarks can be made concerning the limits of validity of these methods. For a hydrostatic load, $P_h = p_i = p_l = p_e$, of large magnitude, the stress-strain relation should no longer be linear, thus making Hooke's law invalid. The work of Bridgman[8] on the compressibility of pure iron shows that, at 12,000 atmospheres, there exists small divergence from linearity.

Furthermore, if the deformations become large, the relations between the components of the deformation tensor and the spatial derivatives of the components of the displacements become quadratic. At this point, the Lamé equations that are formed from the linear forms at these relations are no longer valid, and the relations of elastic loading, which are derived from them, must be entirely reconsidered. Thus, even if the criterion of plasticity used, as in the case for the criteria of Von Mises, Mohr-Caquot and Tresca, implies the condition that a hydrostatic constraint does not cause plastic deformation, it does not automatically result that a hydrostatic load, $p_h = p_i = p_l = p_e$, protects the cylinder from all plastic flow.

Conversely, if the loads, though large, are not isotropic ($p_i \neq p_l \neq p_e$) it can be considered that a plastic law governs the deformation beyond the elastic

[8] Bridgman, P. W., "The Physics of High Pressure," 2nd Edition, Bell and Sons, London, England, 1949, p. 154.

regime of Hooke's law. There is no phase of a nonlinear law elasticity and consequently, the relations of elastic loading shown remain valid.

CONCLUSIONS

The graphical method described allows the resolution of problems relative to elastic loading in a more varied manner than that of the calculations. It allows a better examination of the variables that can be worked on to bring back the end of the vector of load on, or at the interior, of the elastic boundary. Furthermore, using the Tresca criterion makes the graphical construction remarkably easy.

APPENDIX.—NOTATION

The following letter symbols have been adopted for use in this paper:

k = $\frac{r_e}{r_i}$;

L = longitudinal load;

M = k^2;

p_e = external pressure;

p_h = hydrostatic pressure;

p_i = internal pressure;

p_l = $- \frac{L}{\left(r_e^2 - r_i^2\right)}$;

p'_e, p'_i, p'_l = projections of p_i, p_l, p_e onto the plane π;

r, θ, z = cylindrical coordinates;

r_e = external radius;

r_i = internal radius;

V = minor axis of the ellipse;

V_1, V_2, V_3 = components used to define the angle, ψ;

W = major axis of the ellipse;

W_1, W_2, W_3 = components used to define the angle, ψ;

Z = axes pointing in the direction (1, 1, 1);

Z_0 = $\frac{1}{\sqrt{3}}\left(p_i + p_l + p_e\right)$;

σ_c = elastic limit for pure compression;

σ_M = major stress;

σ_m = minor stress;

σ_0 = elastic limit for pure tension;

σ_r = radial stress;

σ_z = longitudinal stress;

σ_θ = circumferential stress;

χ_1, χ_2 = components used to define the angle, ψ;

ψ = angle V, p'_e; and

$\overrightarrow{OP}$ = load vector.

KEY WORDS: cylinders; deformation; engineering mechanics; loads (forces)

ABSTRACT: A thick-walled cylinder submitted to uniformly distributed internal and external pressures and to a uniformly distributed longitudinal load is considered. A graphical construction is established allowing the determination of whether the material does or does not remain elastic under this state of loads, or the selection of the value of one pressure with a view to maximizing another without the cylinder undergoing plastic deformation. Three different constructions are given corresponding to the use of the criteria of Von Mises, Tresca, and of a linearized form of the intrinsic curve of Mohr-Caquot. Several remarks on the conditions and limits in the use of this method are included.

REFERENCE: Epain, Raymond, and Vodar, Boris, "Elastic Loading of High Pressure Cylinders," Journal of the Engineering Mechanics Division, ASCE, Vol. 90, No. EM5, Proc. Paper 4108, October, 1964, pp. 363-374.